Recent Results in Nonlinear Geometry of spaces

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02 et 03 décembre 2015

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Introduction

Un certain nombre de propriétés linéaires des espaces de Banach a été caractérisé en terme non linéaire. L'une des classes la plus importante des opérateurs linéaires entre les espaces de Banach est la classe des opérateurs linéaires p-sommants. Cette classe introduite est développée dans les années soixante dix par A. Grothendieck, A. Pietsch , J. Lindenstrauuss et A. Pełczyński a été identifiée comme le moteur le plus important dans le développement moderne de la théorie des espaces de Banach. Dans cette communication, on présentera les résultats recents correspondant à cette théorie.

Fréchet

On va donner quelques ingredients concernants les espaces métriques. La notion d'espaces métriques a été formalisée par le français Maurice Fréchet dans sa thèse "Doctorat d'Etat" en 1906 sous la direction de Hadamard (voir, "*Sur quelques points du calcul fonctionnel*", Rendic. Circ. Mat. Palermo 22 (1906) 1–74) et il est parmi les premiers qui a utilisé ce mot (ou celui qui l'a introduit). Il a dirigé parmi d'autres les thèses de Nachman Aronszajn et Ky Fan. Il est élu membre de l'Académie polonaise des sciences en 1929, de la Société royale d'Édimbourg en 1947 et de l'Académie des sciences de Paris en 1956. Il est né à Maligny le 2 septembre 1878 et mort à Paris le 4 juin 1973.

Espaces métriques

Définition. Soit X un ensemble non vide. On dit que d est une distance sur X si et seulement si d est une application de X^2 dans $\mathbb{R}^+$ telle que pour tout $(x; y; z) \in X^3$, on a

(i)	$d(x; y) = 0 \Longleftrightarrow x = y$	(séparation),
(ii)	$(x; y) = d(y; x)$	(symétrie),
(iii)	$d(x; z) \leq d(x; y) + d(y; z)$	(inégalité triangulaire).

Soit (X, d_X, e) est un espace métrique pointé (i.e., e un élément neutre ou distingué, on prend 0 si X est normé). On note par
$\mathcal{M}_0 = \{\text{espaces métriques complets pointés}\}$.

Remarques

- *Soit $X \in \mathcal{M}_0$. Il existe une isométrie de X dans $\ell_\infty(X)$.*

Let $X \in M_0$. Let x_0 be in X. Define
$f : X \longrightarrow l_\infty(X)$ by $f(x)(y) = d(x,y) - d(y,x_0)$
$(f(x) = (d(x,y) - d(y,x_0))_{y\in X})$.
Formellement, Kazimierz Kuratowski fut le premier à introduire ce plongement, mais Maurice Fréchet en avait déjà formulé une variante très proche, dans un article où il donna la première définition de la notion d'espace métrique.

- *Every metric space (X,d) is isometric to a subspace of $\mathcal{C}(B_{X^\#})$.*

Define $i_X : X \longrightarrow \mathcal{C}(B_{X^\#})$ by $i(x)(f) = f(x)$.

Les applications lipschitziennes

Le morphisme natural entre les espaces métriques est la fontion lipschitzenne comme les opérateurs linéaires dans les espaces normés.

Définition

Soient (X, d_X) , (Y, d_Y) deux espaces métriques. Une application $f : X \longrightarrow Y$ est dite lipschitzienne, s'il existe $c > 0$ telle que

$$\forall x, y \in X : \quad d_Y(f(x), f(y)) \leq c d_X(x, y) \tag{2.1}$$

On note par

$$\begin{aligned} \mathrm{Lip}(f) &= \sup_{x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \\ &= \inf \{c : \quad \text{vérifiant l'inégalité } (2.1)\} . \end{aligned}$$

Let (X, e_X, d_X), (Y, e_Y, d_Y) be pointed metric spaces. We say a map $f : (X, e_X, d_X) \longrightarrow (Y, e_Y, d_Y)$ preserves distinguished point if $f(e_X) = e_Y$.

Isomorphism

Définition. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective (one-to-one=injective and onto=surjective) and both f, f^{-1} are Lipschitz. In this case X and Y are called Lipschitz isomorphic or Lipschitz homeomorphic (Kalton) = quasi-isometric (Nik Weaver). A bi-Lipschitz function f is an isometry if

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) = d_X(x, y).$$

Let X, Y be (finite) metric spaces, $|X| = |Y|$. The Lipschitz distance between X, Y is

$$d(X, Y) = \inf \left\{ \mathrm{Lip}\,(f)\, \mathrm{Lip}\left(f^{-1}\right) ;\, f \text{ one-one from } X \text{ onto } Y \right\}$$

where $|X|$ denotes cardinal of X.

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- and the Banach -Mazur distance by the Lipschitz distance or distortion.

Espaces rétracts

Définition. Soit X un espace métrique et soit E un sous-espace de X. Une fonction lipschitzienne $p : X \longrightarrow E$ est appelé une rétraction lipschitzienne si $p \diagup E = id_E$. Dans ce cas, on dit que E est un rétracte lipschitzien de X. Un rétract lipschitzien absolu est un espace métrique qui est un rétract de tout espace qui le contient.

Proposition

Soit Y un espace métrique. Alors les propriétés suivant sont équivalentes.

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2. *Pour tout métrique X, pour tout sous-ensemble $E \subset X$ et pour toute fonctions lipschitzienne $f : E \longrightarrow Y$* il existe une extension lipschitzienne $\widetilde{f} : X \longrightarrow Y$.

Proposition

Soit Y un espace métrique. Alors les propriétés suivant sont équivalentes.

1. *L'espace Y est un rétract lipschitzien absolu..*
2. *Pour tout métrique X, pour tout sous-ensemble $E \subset X$ et pour toute fonctions lipschitzienne $f : E \longrightarrow Y$* il existe une extension lipschitzienne $\widetilde{f} : X \longrightarrow Y$.
3. *Pour tout espace métrique Z contenant Y et pour chaque espace métrique F, alors toute fonction lipschitzienne $f : Y \longrightarrow F$* il existe une extension lipschitzienne $\widetilde{f} : Z \longrightarrow F$.

Metric trees

Metric trees are described by differentes names and also given by different definitions (some authors requiered the completion). We give the current definition.

For more details on this section we can consult A. G. Aksoy and T. Oikhberg, *Some results on metric trees.* Preprint, A. G. Aksoy and B. Maurizi.*Metric trees, Hyperconvexhulls, and extensions.*Turkish Math. J. **32** (2008), 219-234, A. G. Aksoy and M. A. Khamsi, *A selection theorem in metric trees.*Proc. Amer. math. Soc. **134** (2006), 2957-2966, Aksoy and M. A. Khamsi. *Fixed points of uniformly lipschitzian mappings in metric trees.* Scientae Mathematicae Japonicae, **65**(1) (2007), 31-41. The study of injective envelopes of metric spaces, also known as metric trees ($\mathbb{R}$-trees or T-theory), has its motivation in many subdisciplines of mathematics as well as biology, medicine and computer science. This concept was introduced at the end of the 1970 by J. Tits (J. Tits. *A Theorem of Lie–Kolchin for trees.*Contributions to algebra (collection of papers dedicated to Ellis Kolchin), Academic Press, New York 1977, 377–388). The equivalent definitions T-theory is due to A. Dress and $\mathbb{R}$-trees is due to J. Tits.

Définitions

Définition 1. Let (X, d) be a metric space and let x, y be in X. A geodesic segmnet from x to y (or metric segment denoted by $[x, y]$) is the image of an isometric embedding $\alpha : [a, b] \longrightarrow M$ of a closed interval $[a, b] \subset \mathbb{R}$ such that $\alpha(a) = x$ and $\alpha(b) = y$. A metric space is called geodesic if any two points can be connected by a metric segment.

Définition 2. A nonempty metric space (X, d) is a metric tree if for all $x, y, z \in X$, we have

1. There exists a unique segment from x to y.
2. If $[x, z] \cap [z, y] = \{z\}$, then $[x, z] \cup [z, y] = [x, y]$.

Définition 3. (Aksoy and. Maurizi**).** A metric tree X is a metric space (X, d) satisfying the following conditions.

1. For every distinct $x, y \in X$, there is a unique isometry $\varphi_{x,y} : [0, d(x, y)] \longrightarrow X$ such that $\varphi_{x,y}(0) = x$ and $\varphi_{x,y}(d(x, y)) = y$.
2. For every one-to-one (injective) continuous mapping $f : [0, 1] \longrightarrow X$, we have

$$d(f(0), f(t)) + d(f(t), f(1)) = d(f(0), f(1)), \quad \forall t \in [0, 1].$$

Remarque

Let (X, d) be a metric space. One say that d

- is ultrametric if it satisfies

$$\forall (x, y, z) \in X^3, d(x, z) \leq \max(d(x, y), d(y, z))(\text{ultrametric inegality})$$

$$d^2(x, y) + d^2(u, v) \leq d^2(x, u) + d^2(y, v) + d^2(x, v) + d^2(y, u).$$

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- satisfies the foor point condition $(4PC)$ if, for any (x, y, u, v) in X we have

$$d(x, y) + \delta(u, v) \leq \max\{d(x, u) + \delta(y, v), d(x, v) + \delta(y, u)\}$$

$$d^2(x, y) + d^2(u, v) \leq d^2(x, u) + d^2(y, v) + d^2(x, v) + d^2(y, u).$$

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- satisfies Reshetnyak's inequality if, for any (x, y, u, v) in X we have

$$d^2(x, y) + d^2(u, v) \leq d^2(x, u) + d^2(y, v) + d^2(x, v) + d^2(y, u).$$

Espace de Lipschitz

Définition. Soit X un espace métrique et Y un espace de Banach. On note par $\mathrm{Lip}(X,Y)$ l'espace de toutes les fonctions lipschitziennes bornées de X dans Y

$$\mathrm{Lip}(X,Y) = \{ \text{ fonctions lipschitziennes bornées } f : X \longrightarrow Y\}$$

muni de la norme

$$\|f\|_{\mathrm{Lip}(X,Y)} = \max\{\|f\|_\infty, \mathrm{Lip}(f)\} \tag{3.1}$$

Si $Y = \mathbb{R}$ alors $\mathrm{Lip}(X,\mathbb{R}) = \mathrm{Lip}(X)$.
$\mathrm{Lip}(X)$ *est un espace de Banach par la norme* (3.1), *pour tout espace métrique* X.

Dual de Lipchitz

Définition (Dual de Lipschitz)

Soit X un espace métrique pointé. On appelle dual de Lipschitz de X et on le note par $X^{\#}$, l'espace de Banach des formes lipschitziennes sur X, i.e.,

$$X^{\#} = \mathrm{Lip}_0\left(X, \mathbb{R}\right) = \mathrm{Lip}_0(X)$$

muni de la norme

$$\mathrm{Lip}(x^{\#}) = \sup_{x \neq y} \frac{\left| x^{\#}\left(x\right) - x^{\#}\left(y\right) \right|}{d_X\left(x, y\right)}.$$

Définition. Soient $X, Y \in \mathcal{M}_0$ et $g : Y \longrightarrow X$ une fonction lipschitzienne qui préserve le point distingué. Nous définissons la composition d'application

$$\begin{array}{rccl} C_g : & \mathrm{Lip}_0(X) & \longrightarrow & \mathrm{Lip}_0(Y) \\ & f & \longmapsto & C_g(f)\ (C_g\,(f)\,(x) = f \circ g(x)). \end{array}$$

Proposition

Soient $X, Y \in \mathcal{M}_0$ et $g : Y \longrightarrow X$ une fonction lipschitzienne qui préserve le point de base. Alors C_g est un application linéaire borné et $\|C_g\| = \mathrm{Lip}(g)$.

We shall present first the construction of Arens and Eels (R. F. Arens and J. Eels, *On embedding uniform and topological spaces*, Pacific J. Math **6** (1956), 397-403. (see also Weaver (N. Weaver, *Lipschitz Algebras*, World Scientific, Singapore 1999) of the space for which $\mathrm{Lip}_0(X)$ is the dual space. Remark that another, less explicit, realization of $\mathrm{Lip}_0(X)$ as a dual space was given by K. de Leeuw,
Banach space of Lipschitz functions, Studia Math. **21** (1961),55-66 (see also Weaver). It was shown by Arens and Eels and Weaver that $\mathrm{Lip}_0(X)$ is even a dual Banach space, i.e., there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z.

This canonical space is known as the Arens-Eells space in T. Fiegiel and N. Tomczack-Jaegermann, *Projections onto hilbertian subspaces of Banach spaces*, Israel J. Math. **33** (1979), 155-171. and the Lipschitz-free space on X in N. J. Kalton, *Spaces of Lipschitz and Hölder functions ant their applications*, Collect. Math. **55**(2) (2004), 171-217. It well be noted as in by $\mathcal{F}(X, d_X)$.

Proposition

Soit (X, d_X) un espace métrique. Munissons l'espace des molécules $M(X)$ de la norme suivante

$$\|m\|_{M(X)} = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \, d_X(x_i, x_i^{'}) : m = \sum_{i=1}^{n} \lambda_i \left(\mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{x_i^{'}\}} \right) . \right\}$$

Alors notons $Æ(X)$ de la complètè de l'espace normé $(M(X),), \|.\|_{M(X)})$.et parfois l'espace $Æ(X)$ s'appelle l'espace de *Lipschitz libre* de X (voir N. J. Kalton, *Spaces of Lipschitz and Hölder functions ant their applications,* Collect. Math. **55**(2) (2004), 171-217.) et on le note aussi par $\mathcal{F}(X)$.

L'espace Æ(X)

Denote by $Æ(X, d_X)$ the completion of the normed space $(M(X), \|.\|_{M(X)})$. This space was first introduced by Arens and Eells in 1956. Originally, the basic idea goes back to L. V. Kantorovich, *On the translocation of masses*, Dokl. Akad. Nauk SSSR **37** (1942), 227-229.. The terminology Arens-Eells space $Æ(X, d_X)$ is due to Weaver. A different notation was used in (N. J. Kalton and G. Godefory, *Lipschitz-free Banach spaces*, Studia Math. **69** (2003), 121-141) by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X, d_X)$
Théorème. *Soit X un espace métrique pointé (i.e., $X \in M_0$.). Alors*

$$Æ(X)^* = \mathrm{Lip}_0(X).$$

Lemma. Si X est un espace de Banach alors il existe une projection P de $X^{\#}$ dans X^{*} (voir Y. Benyamini and J. Lindenstrauuss, *Geometric nonlinear functional analysis*,Vol. 1, Amer. math. Soc. Colloq. Pub. **48** Amer. Math. Soc. RI, 2000 et Kalton, Godefory pour plus de détails).

Corollaire

Soit X un espace métrique pointé.

1. *Pour toute molécule m, nous avons*

$$\|m\|_{Æ} = \sup_{f \in B_{X^\#}} |\langle m, f \rangle| .$$

2. $\|.\|_{Æ}$ *est une norme sur l'espace de molécules qui satisfait*

$$\forall x, y \in X : \left\|1_{\{x\}} - 1_{\{y\}}\right\|_{Æ} = d_X(x, y).$$

3. $\|.\|_{Æ}$ *est la plus grande semi-norme sur l'espace de molécules qui satisfait*

$$\forall x, y \in X : \left\|1_{\{x\}} - 1_{\{y\}}\right\|_{Æ} \leq d_X(x, y).$$

Linéarisation

Remarque. Soit (X, d_X) un espace métrique. L'application $i_X : X \longrightarrow Æ(X)$ définie par

$$i_X\,(x) = \left(1_{\{x\}} - 1_{\{e\}}\right)$$

Remarque. Soit (X, d_X) un espace métrique. L'application $i_X : X \to Æ(X)$ définie par

$$i_X\,(x) = \left(1_{\{x\}} - 1_{\{e\}}\right)$$

est un plongement isométrique de X dans $Æ(X)$.

Théorème (Weaver)

Soient X un espace métrique pointé, Y un espace de Banach et $T : X \longrightarrow Y$ *une fonction lipschitzienne* qui préserve le point de base (i.e., $f(e) = 0$). Alors il existe un unique opérateur linéaire borné $u =: Æ(X) \to Y$ tel que $T = u \circ i_X$ et $\|u\| = \mathrm{Lip}(T)$.

Graphe

$$\begin{array}{ccc} \text{Æ}(X) & & \\ i_X \uparrow & \overset{u}{\searrow} & \\ X & \overset{T}{\longrightarrow} & Y \end{array}$$

Comments 1

1-The Arens Eells space $\text{Æ}(X)$ is isomorphic to $L_1(\mathbb{R})$ for all finite dimensional space X and $\text{Æ}(l_1^n)$ is isometrically isomorphic to $L_1(\mathbb{R})$. $\text{Æ}(l_1)$ is isometrically isomorphic to $L_1(\mathbb{R})$. M. Dubeia, E.D. Tymchatynb, A. Zagorodnyuka, *Free Banach spaces and extension of Lipschitz maps,* Topology **48** (2009), 203-212.

2- If (X, ρ) is a discrete metric space, then $\text{Æ}(X)$ is isomorphic to $l_1(X)$.

3- If X is a Banach space, then X is 1-Lipschitz retract of $\text{Æ}(X)$.

4- Let X be a separable Banach space. Then every separable Banach space E is isomorphic to a quotient of $\text{Æ}(X)$.

5- Let X be a Banach space, then $\text{Æ}(X) = \text{Æ}(B_X)$.

6- Let X be a subset of $\mathbb{R}^n$ such that $\overset{\circ}{X} \neq \emptyset$. Then, $\text{Æ}(X) = \text{Æ}(\mathbb{R}^n)$.

Comments 2

A. Naor, G. Schechtman proved in [A. Naor, G. Schechtman, *Planar Earthmover is not in L_1*, SIAM J. Comput. **37** (3) (2007), 804-826] that $\text{Æ}(\mathbb{R}^2)$ cannot be seen as a subspace of any L_1 if $\dim(X) > 1$. A. Godard proved in [A. Godard, *Tree metrics and their Lipschitz-free spaces*, Proc. Amer. Math. Soc. **138** (12) (2010), 4311-4320] that $\text{Æ}(X)$ is isometric to a subspace of an $\mathcal{L}_1$-space if, and only if, X embeds isometrically into an $\mathbb{R}$-tree. Godard's theorem for metric tree is the following: Let M be a pointed metric space. Then $\text{Æ}(M)$ is isometric to a subspace of L_1 if and only of embeds isometrically in a finite metric tree.

Définition et propriété

On définit un autre opérateur adjoint qui est appellé adjoint d'opérateur lipschitizien.
Soit X un espace métrique pointé et Y un Banach réel, Sawashima (I. Sawashima, *Methods of Lipschitz duals*, in Lecture Notes Ec. Math. Sust, **419**, Springer Verlag (1975), 247-259) a défini l'adjoint lipschitzien $T^{\#} : Y^{\#} \longrightarrow X^{\#}$ d'une application lipschitzienne $T \in \mathrm{Lip}_0(X, Y)$ par la formule

$$\begin{array}{rccl} T^{\#}: & Y^{\#} & \longrightarrow & X^{\#} \\ & g & \longmapsto & T^{\#}g = g \circ T \text{ (i.e., } \langle T(x), g\rangle = \langle x, T^{\#}(g)\rangle \text{).} \end{array}$$

Soit $T \in \mathrm{Lip}_0(X, Y)$*. L'opérateur* $T^{\#}$ *est linéaire et*
$\|T^{\#}\| = \mathrm{Lip}(T) = \|T^{\#}/Y^{*}\|$.

Conclusion

Banach spaces	metric spaces
isometric isomorphism	isometric
topological isomorphism	bi-Lipschitz or quasi-isometric
Banach-Mazur distance	Lipschitz distance or distortion
Topological dual	Lipschitz dual

Introduction

The nonlinear version of p-summing operators was introduced by J. D. Farmer and W. B. Johnson in (J.D.Farmer and W.B. Johnson, *Lipschitz p-summing operators*, Proc. Amer. Math. Soc. **137**(9) (2009), 2989-2995). They called it Lipschitz p-summing operator. Let (X, d_X) be a pointed metric space, i.e., a metric space $(X, d_X)$with a distinguished element noted 0 and let Y be a Banach space. We denote by $X^{\#} = \mathrm{Lip}_0(X, \mathbb{R}) = \mathrm{Lip}_0(X)$ the space of all Lipschitz mappings $f : X \longrightarrow \mathbb{R}$ vanishing at 0. The space $(X^{\#}, \mathrm{Lip}(\cdot))$ equipped with the norm $\mathrm{Lip}_0(\cdot)$ is a Banach space and $B_{X^{\#}}$ (here $B_{X^{\#}}$ is the closed unit ball of $X^{\#}$) is a compact Hausdorff space in the topology of pointwise convergence on X ($B_{X^{\#}}$ it is unwieldy).

Definition

A Lipschitz map $T : X \to Y$ is called Lipschitz p-summing $(1 \leq p < \infty)$, if there is a positive constant C such that for all $n \in \mathbb{N}$, $\{x_i\}_{1 \leq i \leq n}, \{y_i\}_{1 \leq i \leq n}$ in X and all $\{a_i\}_{1 \leq i \leq n} \subset \mathbb{R}^+$, we have

$$\sum_{i=1}^{n} a_i d_Y\left(T(x_i), T(y_i)\right)^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(y_i)|^p. \tag{6.1}$$

The Lipschitz p-summing $(1 \leq p < \infty)$norm, $\pi_p^L(T)$ of T is the smallest constant C verifying (6.1).The space $\pi_p^L(X, Y)$of Lipschitz p-summing functions from any metric space into Y is a Banach space under the norm $\pi_p^L(.)$. If T is linear then $\pi_p^L(T) = \pi_p(T)$.
Notice that the definition is the same if we restrict to $a_i = 1$ (we find this implicitly in (J. D. Farmer and W. B. Johnson, *Lipschitz p-summing operators*, Proc. Amer. Math. Soc. **137**(9) (2009), 2989-2995)). Let $i : Y \longrightarrow Z$ be an isometry betwen Banach spaces, we have $\pi_p^L(i \circ T) = \pi_p^L(T)$. Also, $\pi_p^L(T) = \sup \pi_p^L(T\mid_K)$, the supremum being over all finite subsets K of X. **it is a good generalization**.

Theorem

Let X *be a metric space and* Y *be a Banach space. Consider* $1 \leq p < \infty$. *The following properties are equivalent for a mapping* $T : X \longrightarrow Y$ *and a positive constant* C.

(a) The mapping T *is Lipschitz p-summing and* $\pi_p^L(T) \leq C$.

(b) There is a RPM μ *on* $B_{X^\#}$ *such that for all* x, y *in* X, *we have*

$$\|T(x) - T(y)\| \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f) \right)^{\frac{1}{p}}.$$

(c) For any isometric embedding j *of* Y *into a* 1*-injective space* Z, *the following diagram commute*

$$\begin{array}{ccc} L_\infty(B_{X^\#}, \mu) & \overset{i_p}{\longrightarrow} & L_p(B_{X^\#}, \mu) \\ i \uparrow & & \downarrow \widetilde{T} \\ X \overset{T}{\longrightarrow} Y & \overset{j}{\longrightarrow} & Z \end{array}$$

with $Lip(\widetilde{T}) \leq C$.

Nonlinear version of GT

Théorème. (Grothendieck's theorem) (J. D. Farmer and W.B. Johnson), D. Chen and B. Zheng, *Remarks on Lipschitz p-summing operators*, Proc. Amer. Math. Soc. **139** (8), (2011), 2891-2898. and K. Saadi, *Some properties of Lipschitz strongly p-summing operators*, J. Math. Anal. Appl. **423** (2015), 1410-1426) Let X be pointed metric space such that X embeds isometrically into an $\mathbb{R}$ -tree. Then for any Hilbert space H , we have

$$\pi_1^L(X, H) = \mathrm{Lip}_0(X, H)$$

and

$$\pi_1^L(T) \leq K_G \mathrm{Lip}(T) \textit{ for evry } T \textit{ in } \mathrm{Lip}_0(X, H).$$

Definition (Mezrag, Tallab)

The following definition was studied by X. Mujica in (X. Mujica, *$\tau(p; q)$-summing mappings and the domination theorem*, Portugal. Math. (N. S.) **65** (2) (2008), 211–226) for multilinear operators, which generalizes absolutely τ-summing linear operators introduced by A. Pietsch in (A. Pietsch, Operator ideals. North-Holland Math. Library 20, North Holland Publishing Co., Amsterdam 1980). This generalization is due to Mezrag and Tallab (called integrale operator by Paulette Saab!).

Definition

Let T be in $\mathrm{Lip}_0(X, E)$ and consider $1 \leq q \leq p < \infty$. We say that T is Lipschitz $\tau(p, q)$-summing if there is a positive constant C such that, for all $n \in \mathbb{N}$; $(x_i), (x_i') \subset X$; $(a_i^*) \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$, we have

$$\left(\sum_{i=1}^{n} \lambda_i \left| \langle T(x_i) - T(x_i'), a_i^* \rangle \right|^p \right)^{\frac{1}{p}} \leq C \sup_{\substack{\|f\| \leq 1 \\ \|a\| \leq 1}} \left(\sum_{i=1}^{n} \lambda_i \left| (f(x_i) - f(x_i')) \langle a_i^*, a \rangle \right|^q \right)^{\frac{1}{q}}$$

where $f \in X^{\#}$ and $a \in E$. We will denote this class of mappings by $\Pi^{L}_{\tau(p,q)}(X, E)$ and we equip it with the norm $\pi^{L}_{\tau(p,q)}(T) = \inf C$, for the constants that appear in the above expression, for which it becomes a Banach space. When $p = q$, we write $\Pi^{L}_{\tau(p)}$ and $\pi^{L}_{\tau(p)}$ instead of $\Pi^{L}_{\tau(p,p)}$ and $\pi^{L}_{\tau(p,p)}$ respectively and we say that T is Lipschitz $\tau(p)$-summing.

Domination theorem 1

Theorem

Consider $T \in \mathrm{Lip}_0(X, E)$ *and a positive constant.*
(1) *The operator* T *is Lipschitz* $\tau(p)$*-summing and* $\pi^L_{\tau(p)}(T) \leq C$.
(2) *There exist Radon probability measures* μ_1 *on* $B_{X^\#}$ *and* μ_2 *on* $B_{E^{**}}$*, such that for all* x, x' *in* X *and* a^* *in* E^**, we have*

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ \leq \quad & C\left(\int_{B_{X^\#}} \int_{B_{E^{**}}} |(f(x) - f(x')\langle a^*, a^{**}\rangle|^p \, d\mu_1(f) d\mu_2(a^{**})\right)^{\frac{1}{p}}). \end{aligned}$$

Moreover, in this case

$$\pi^L_{\tau(p)}(T) = \inf\{C > 0 : \text{ for all } C \text{ verifying the last inequality}\}.$$

Achour et Cie, Saadi

The following notion was introduced independently by (K. Saadi, *Some properties of Lipschitz strongly p-summing operators*, J. Math. Anal. Appl. **423** (2015), 1410-1426) and (R. Yahi, D. Achour and P. Rueda, *Absolutely summing Lipschitz conjugates*, Mediterr. J. Math. (2015), 1-13)). For our convenience, we will adopt the notation of Yahi,. Achour and Rueda.

Definition

A Lipschitz map $: X \to E$ is Lipschitz strongly p-summing $(1 < p \leq \infty)$ if there is a constant $C > 0$, such that for all $n \in \mathbb{N}$, $(x_i)_{1 \leq i \leq n}$, $(x_i')_{1 \leq i \leq n}$ in X, $(a_i^*)_{1 \leq i \leq n}$ in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in $\mathbb{R}_+$, we have

$$\sum_{i=1}^{n} \lambda_i \left| \langle T(x_i) - T(x_i'), a_i^* \rangle \right| \leq C \left(\sum_{i=1}^{n} \lambda_i d_X(x_i, x_i')^p \right)^{\frac{1}{p}} \omega_{p^*}((a_i^*)_i).$$

We denote by $\mathcal{D}^{L}_{st,p}(X,E)$ the class of all Lipschitz strongly p-summing operators from X into E and $d^{L}_{st,p}(T)$ the smallest C such that inequality (7.3) holds. This generalizes the definition introduced by (J. S. Cohen *Absolutely p-summing, p-nuclear operators and their conjugates*. Math. Ann. **201** (1973), 177-200) in the linear case. If T is linear, then in the absence of $B_{X^{\#}}$ we have $\mathcal{D}^{L}_{st,p}(X,E) = \mathcal{D}_{p}(X,E)$.

Domination theorem

Now, we give the domination theorem of the strongly Lipschitz p-summing (see Saadi and Yahi, Achour, Rueda).

Theorem

A Lipschitz operator T from X into E is Lipschitz strongly p-summing $(1 < p < \infty)$ if, and only if, there exist a positive constant C and Randon probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, we have

$$\left|\left\langle T(x) - T(x'), a^* \right\rangle\right| \leq C d_X\left(x, x'\right)\left(\int_{B_{E^{**}}} \left|a^*\left(a^{**}\right)\right|^{p^*} d\mu\left(a^{**}\right)\right)^{\frac{1}{p^*}}.$$

Moreover, in this case

$$d_{st,p}^L\left(T\right) = \inf\left\{C > 0 : \textit{ for all } C \textit{ verifying the precedent inequality}\right\}.$$

We introduce the following generalization to Lipschitz operators of the class of Cohen p-nuclear operators studied in Cohen. It is a particular case from that defined by J. A. Chàvez-Domènguez which called the Lipschitz (r, p, q)-summing operators if we take $(r, p, q) = (1, p, p^*)$ and $k_i = 1$ for all i. The notion of p-nuclear operators was introduced by A. Person and A. Pietsch in, (A. Person and A. Pietsch *p-nukleare and p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969), 213-222). Initially the definition of nuclear operators for Banach spaces, was given by Grothendieck in (A. Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques*" Bol. Soc. Mat. São Paulo **8** (1956), 1-79).

J. S. Cohen has initiated another concept of p-nuclear operators which is not the same as the precedent notion and was generalized to (p, q)-nuclear operators $(1 \leq q \leq \infty)$ by H. Apiola in (H. Apiola, *Duality between spaces of p-summable sequences, (p, q)-summing operators and characterizations of nuclearity*, Math. Ann. **219** (1976), 53–64). In (D. Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators*, Nonlinear Analysis **75** (2012), 5270-5282), D. Chen and B. Zheng has generalized this notion to Lipschitz operators. For distinguish these two notions, we say Cohen p-nuclear operators for that investigated by J. S. Cohen and we try to generalize this notion to Lipschitz operators.

Definition (Mezrag Tallab)

Definition

A Lipschitz operator $T : X \longrightarrow E$ is Cohen Lipschitz p-nuclear ($1 < p < \infty$), if there is a positive constant C such that for any n in $\mathbb{N}$; $(x_i)_{1 \leq i \leq n}, (x_i')_{1 \leq i \leq n}$ in X; $(a_i^*)_{1 \leq i \leq n}$ in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in $\mathbb{R}_+$, we have

$$\left| \sum_{i=1}^{n} \lambda_i \left\langle T(x_i) - T(x_i'), a_i^* \right\rangle \right|$$

$$\leq \quad C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n} \lambda_i \left| f(x_i) - f(x_i') \right|^p \right)^{\frac{1}{p}} \sup_{\|a\|_E \leq 1} \left(\sum_{i=1}^{n} \left| \left\langle a, a_i^* \right\rangle \right|^{p^*} \right)^{\frac{1}{p^*}}.$$

The smallest constant C which is noted by $\eta_p^L(T)$, such that the above inequality holds, is called the Cohen Lipschitz p-nuclear norm on the space $\mathcal{N}_p^L(X, E)$ of all Cohen Lipschitz p-nuclear operators from X into E which is a Banach space. For $p = 1$ and $p = \infty$ we have like the linear case $\mathcal{N}_1^L(X, E) = \Pi_1^L(X, E)$ and $\mathcal{N}_\infty^L(X, E) = \mathcal{D}_{st,\infty}^L(X, E)$.

Theorem

Consider $T \in \mathrm{Lip}_0(X,E)$ and C a positive constant. Then the following assertions are equivalent.

(1) *The operator T is Cohen Lipschitz p-nuclear and $\eta_p^L(T) \leq C$.*

(2) *For any n in $\mathbb{N}$; $(x_i)_{1\leq i\leq n}$, $(x_i')_{1\leq i\leq n}$ in X; $(a_i^*)_{1\leq i\leq n}$ in E^* and $(\lambda_i)_{1\leq i\leq n}$ in $\mathbb{R}_+$, we have*

$$\sum_{i=1}^{n} \lambda_i \left| \langle T(x_i) - T(x_i'), a_i^* \rangle \right|$$

$$\leq C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} \lambda_i \left| f(x_i) - f(x_i') \right|^p \right)^{\frac{1}{p}} \sup_{\|a\|_E\leq 1} \left(\sum_{i=1}^{n} \left| \langle a, a_i^* \rangle \right|^{p^*} \right)^{\frac{1}{p^*}}.$$

Theorem

(3) *There exist Radon probability measures* μ_1 *on* $B_{X^\#}$ *and* μ_2 *on* $B_{E^{**}}$, *such that for all* x, x' *in* X *and* a^* *in* E^*, *we have*

$$\begin{aligned} & |\langle T(x) - T(x'), a^* \rangle| \\ \leq \quad & C\left(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu_1(f)\right)^{\frac{1}{p}} \left(\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2(a^{**})\right)^{\frac{1}{p^*}}. \end{aligned}$$

Moreover, in this case
$\eta_p^L(T) = \inf \{ C > 0 :$ *for all* C *verifying the inequality (3.5)*$\}$.

Introduction

In the mid way between continuous linear operators and absolutely summing operators, a scale of linear operators, namely (p, σ)-absolutely continuous operators ($1 \leq p < \infty, 0 \leq \sigma < 1$), were defined by Matter (U. Matter, *Absolutely continuous operators and super-reflexivity*, Math. Nachr. 130 (1987), 193-216) by applying an interpolative ideal procedure. The interpolated operator ideal $\Pi_{p,\sigma}$ of all (p, σ)-absolutely continuous operators was defined as an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators, and shares similar properties with absolutely p-summing operators.

Definition

(Achour et Cie) Let $1 \leq p < \infty$, and $0 \leq \sigma < 1$. Let X be a pointed metric space and E be a Banach space. A mapping $T \in Lip_0(X, E)$ is called (p, σ)-*absolutely Lipschitz* if there exist a Banach space F and a Lipschitz operator $S \in \Pi_p^L(X, F)$such that
$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x)^{\sigma}$ for all $x, x' \in X$.

The space $\Pi_{p,\sigma}^L(X, E)$ is a Banach space.

Pietsch's domination theorem

Theorem

Let $1 \leq p < \infty$, $0 \leq \sigma < 1$ *and* $T \in Lip_0(X, E)$. *The following statments are equivalente.*

1. $T \in \Pi^L_{p,\sigma}(X, Y)$.

2. There is a constant $C \geq 0$ *and a Borel probability measure* μ *on* $B_{X^\#}$ *such that*

$$\|T(x) - T(x')\| \leq C \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x, x')^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu \right)^{\frac{1-\sigma}{p}}$$

for all $x, x' \in X$.

Theorem

3. There is a constant $C \geq 0$ *such that for all* $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and all* $(a_i)_{i=1}^n \subset \mathbb{R}^+$ *we have*

$$\left(\sum_{i=1}^n a_i \| T(x_i) - T(x_i') \|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$
$$\leq \quad C \sup_{f \in B_{X\#}} \left(\sum_{i=1}^n a_i \left(|f(x_i) - f(x_i')|^{1-\sigma} d(x_i, x_i')^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

Furthermore, the infimum of the constants $C \geq 0$ *in* (2) *and* (3) *is* $\pi_{p,\sigma}^L(T)$.

Ideal property

Proposition

Let X, Y, X_0, Y_0 *be metric spaces. If* $v : X_0 \longrightarrow X$, $w : Y \longrightarrow Y_0$ *are Lipschitz mappings and* $T : X \longrightarrow Y$ *is* (p, σ)*-absolutely Lipschitz. Then* wTv *is* (p, σ)*-absolutely Lipschitz and*

$$\pi^L_{p,\sigma}(wTv) \leq Lip(w) Lip(v) \pi^L_{p,\sigma}(T).$$

Introduction

L'idéal au sens de pietsch a été généralisé par plusieurs auteurs indépendament et en même temps. Parmi tant d'autres,

1. Achour, Rueda, Sanchez-Perez, and Yahi, "*Lipschitz operator ideals and the approximayion property*", Apr 2014, publié
2. Gabrrera-Padilla, Chavez-Dominguez, Jimenez-Vargas and Villegas-Vallecillos, "*Duality for for ideals of Lipschitz maps*", 19 Jun 2015.
3. Manaf Adnan Saleh saleh, "*Nonlinear operators ideals betweenmetric spaces and Banach spaces* PARTI" 03 Jul 2015.
4. Saadi, *"On the composition ideals of Lipschitz mappings"* 21 Jul 2015.
5. Aussi, J. A. Chavez-Dominguez dans sa thèse 2012 texas university "operator ideals in lipschitz and operator spaces theory.

Mais, on pense que le premier article est le plus complet en conformité avec toutes les définitions de sommabilité.

Conclusion

Lipschitz nuclear operators
Lipschitz integrale operators
Tensor product
Compact operators
Lipscitz p-concave et p-convexe
metric type et cotype
Lipschitz factorization by hilbert space

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